Math 255 Honors: Gram-Schmidt Orthogonalization on the Space of Polynomials

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Abstract

Gram-Schmidt Orthogonalization is a process to construct orthogonal vectors from some basis for a vector space. In this paper we will discuss the Gram-Schmidt process on the set of all polynomials with degree N, use the Gram-Schmidt process to generate the Legendre Polynomials, using Mathematica code, in their normalized and unnormalized forms.

1 Space Axioms

1.1 Vector Space Axioms

Any polynomial must be in the form $\sum_{n=0}^{N} c_n x^n$, for constants $c_0, \ldots, c_n$, where it is assumed that the polynomial is a function of some real paramater x. It is clear that the following axioms are satisfied, due to the properties and definitions of addition and multiplication in $\mathbb{R}$. For any polynomials $p(x)$, $q(x)$, $r(x)$, and scalars $c, d \in \mathbb{R}$.

1. Addition Axioms
   (a) $p + q = q + p$.
   (b) $(p + q) + r = p + (q + r)$.
   (c) $0 + p = p + 0 = p$.
   (d) $(-p) + p = 0$.
2. Scalar Multiplication Axioms
   (a) $0 \cdot p = 0$
   (b) $1 \cdot p = p$
   (c) $(cd)p = c(dp)$
3. Distributive Axioms
   (a) $c(p + q) = cp + cq$
   (b) $(c + d)p = cp + dp$

Since the axioms are satisfied, the space of such polynomials is a vector space.

1.2 Inner Product Space Axioms

We can define the inner product between two polynomials, $\langle p, q \rangle$ in terms of a real integral:

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx$$

The following axioms are satisfied, again with polynomials p, q, r, and real values c, d.

1. $\langle p, q \rangle = \langle q, p \rangle$, since $p(x)q(x) = q(x)p(x)$ within the integral.
2. $\langle cp, q \rangle = \int_{-1}^{1} cp(x)q(x)dx = c \langle p, q \rangle$.
3. $\langle p + r, q \rangle = \int_{-1}^{1}(p(x)q(x) + r(x)q(x))dx = \langle p, q \rangle + \langle r, q \rangle$.
4. $\langle p, p \rangle = \int_{-1}^{1} p(x)^2 dx > 0$.

So, the given definition satisfies all the properties of an inner product. We will also make use of the definition:

$$\|A\|^2 = \langle A, A \rangle$$

Now that we've established the inner product and vector space, we can define Gram-Schmidt Orthogonalization.

2 Gram-Schmidt Process

2.1 Definition

The Gram-Schmidt process is defined as follows. Assume that S_n is a basis with N elements for the set of polynomials of degree less than or equal to N. The Gram-Schmidt process creates one list of orthogonal vectors, w_n.

1. let $w_0 = S_0$, $e_0 = w_0/\sqrt{\langle w_0, w_0 \rangle}$.
2. Then, define $w_1 = S_1 - w_0 \frac{\langle S_1, w_0 \rangle}{\langle w_0, w_0 \rangle}$.
3. And in general, let $w_n = S_n - \sum_{i=0}^{n-1} w_n \frac{\langle S_n, w_i \rangle}{\langle w_i, w_i \rangle}$.

2.2 Properties

Theorem 1. *A nonzero vector A is orthogonal to a finite set of orthogonal vectors S if and only if it can be obtained from the Gram-Schmidt Process, from some vector C. (ie, $A = C - \langle C, S_1 \rangle / \|S_1\|^2 S_1 - \ldots - \langle C, S_n \rangle / \|S_n\|^2 S_n$, for some C in the vector space).*

Proof. $\leftarrow$) If A is obtained from the Gram-Schmidt process from some vector C, and is nonzero, then for any vector $S_n \in S$, where S has N elements,

$$\begin{aligned}
\langle A, S_n \rangle &= \left\langle C - \sum_{i=1}^{N} S_i \langle C, S_i \rangle / \|S_i\|^2, S_n \right\rangle & (1) \\
&= \langle C, S_n \rangle - \sum_{i=1}^{N} \left\langle S_i \langle C, S_i \rangle / \|S_i\|^2, S_n \right\rangle & (2) \\
&= \langle C, S_n \rangle - \sum_{i=1}^{N} \langle C, S_i \rangle / \|S_i\|^2 \langle S_i, S_n \rangle & (3) \\
&= \langle C, S_n \rangle - \langle C, S_n \rangle / \|S_n\|^2 \langle S_n, S_n \rangle & (4) \\
&= \langle C, S_n \rangle - \langle C, S_n \rangle / \|S_n\|^2 \|S_n\|^2 & (5) \\
&= \langle C, S_n \rangle - \langle C, S_n \rangle & (6) \\
&= 0 & (7)
\end{aligned}$$

Where (3) follows from the definition of a set of orthogonal vectors; $\langle S_i, S_j \rangle = 0$ whenever $i \neq j$.

Since $\langle A, S_n \rangle = 0$, and since n was arbitrary, this means A is orthogonal to every vector in S.

$\rightarrow$) If A is nonzero vector orthogonal to every S_i, then letting $C = A$, applying the Gram-Schmidt process to C yields:

$$\begin{aligned}
& C - \sum_{i=1}^{N} S_i \langle S_i, C \rangle / \|S_i\|^2 & (8) \\
=\ & C - \sum_{i=1}^{N} S_i \cdot 0 / \|S_i\|^2 & (9) \\
=\ & C & (10) \\
=\ & A & (11)
\end{aligned}$$

□

So, since a vector produced from one step of the Gram-Schmidt orthogonalization process will be orthogonal to all vectors previous to it, using the process with $\{S_n\}$ as an input gives you the orthogonal basis $\{w_n\}$ as an output. The above theorem also implies that if you let S be a completely arbitrary basis for the space, then applying the Gram-Schmidt process gives you the general form of all possible orthogonal basis vectors.

3 Legendre Polynomials

The most straightforward basis for the space of polynomials degree less than or equal to N is $S = \{1, x, x^2, \ldots, x^N\}$, but, for example, $\langle 1, x^2 \rangle = \int_{-1}^{1} x^2 dx = ((1)^3 - (-1)^3)/2 = 2/3$, so S is not an orthogonal basis.

However, if we apply the Gram-Schmidt orthogonalization process to the set, letting $N = 2$ for example, we wind up with:

1. $w_0 = 1$.
2. $w_1 = x - 1 \langle 1, x\rangle / \langle 1, 1\rangle = x - 0 = x$.
3. $w_2 = x^2 - 1 \left\langle 1, x^2\right\rangle / \langle 1, 1\rangle - \left\langle x, x^2\right\rangle / \langle x, x\rangle$. Since the rightmost term is the integral from -1 to 1 of an odd function, it's zero. The middle term evaluates to 1/3, and so $w_2 = x^2 - 1/3 = 1/3(3x^2 - 1)$.

Continuing to higher N, we end up with multiples of a family called the Legendre polynomials.

4 Legendre Polynomials Uniqueness

A simple orthogonal basis could be in the form $S = \{c_{(0,0)}, c_{(0,1)}+c_{(1,1)}x, \ldots, \sum_{i=0}^{N} c_{(i,N)}x^i\}$, for some real constants $\{c_{(i,j)}\}$ If we orthogonalize this basis, then as above, up to $N = 1$:

1. $w_0 = c_{(0,0)}$.
2. $w_1 = c_{(0,1)} + c_{(1,1)}x - c_{(0,0)} \left\langle c_{(0,0)}, c_{(0,1)} + c_{(1,1)}x\right\rangle / (2c_{(0,0)}^2)$
   $= c_{(0,1)} + c_{(1,1)}x - 1/(2c_{(0,0)}) \int_{-1}^{1} (c_{(0,0)}c_{(0,1)} + c_{(0,0)}c_{(1,1)}x)dx$
   $= c_{(0,1)} + c_{(1,1)}x - c_{(0,1)}$
   $= c_{(1,1)}x$

The integrals can take long to evaluate, but we find that each w_i is just some multiple of the Legendre polynomials found in the last section, and that the constants $c_{(0,1)}$, $c_{(0,2)}$, $c_{(1,2)}$ etc. disappear from the final result. The following is a mathematica-generated table continuing the process up to $N = 4$.

4.1 Table 1

Table of Legendre Polynomials, generated by the code in appendix B:

N	Gram-Shmidt Generated	Standard	Normalized
0	$c_{(0,0)}$	1	$\frac{1}{\sqrt{2}}$
1	$xc_{(1,1)}$	x	$\sqrt{\frac{3}{2}}x$
2	$\frac{1}{3}\left(3x^2 - 1\right) c_{(2,2)}$	$\frac{1}{2}\left(3x^2 - 1\right)$	$\frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)$
3	$\frac{1}{5}x\left(5x^2 - 3\right) c_{(3,3)}$	$\frac{1}{2}x\left(5x^2 - 3\right)$	$\frac{1}{2}\sqrt{\frac{7}{2}}x\left(5x^2 - 3\right)$
4	$\frac{1}{35}\left(35x^4 - 30x^2 + 3\right) c_{(4,4)}$	$\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$	$\frac{3}{8\sqrt{2}}\left(35x^4 - 30x^2 + 3\right)$

This shows that the Legendre polynomials are unique in that any orthogonal basis consisting of polynomials of increasing degree, starting at a 0th degree polynomial, will just be multiples of the Legendre polynomials. To uniquely determine each polynomial, we can add an extra N equations. The standard Legendre Polynomials are constructed as in the table above, with the added condition that

$$\langle P_n, P_n\rangle = \frac{2}{2n+1}$$

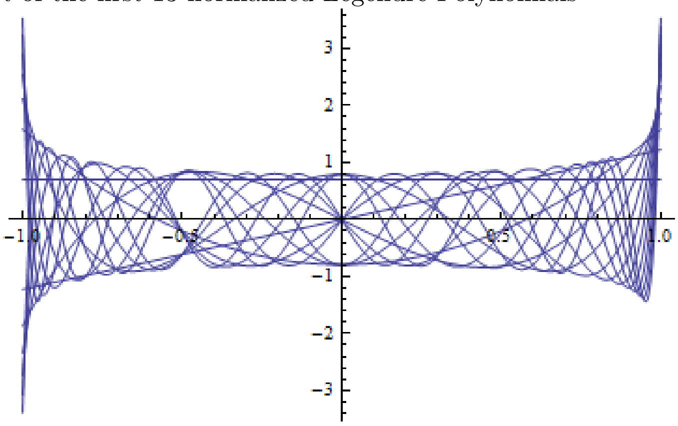
The normalized Legendre Polynomials are constructed by specifying that

$$\langle P_n, P_n \rangle = 1$$

.

4.2 Figure 1

Plot of the first 13 normalized Legendre Polynomials



5 Appendix

A Orthogonalization Code

```
(* Create a polynomial from a list of coefficients *)
polynomial[clist_, x_] := Sum[clist[[n]] x^(n-1), {n, 1, Length[clist]}];

(* Inner product function - Integrates the function from x=-1 to x=1 *)
pInner[p1_, p2_] := Integrate[Function[x, p1 p2][x], {x, -1, 1}];

(* Gram-Schmidt Orthogonalization Code *)
orthogonalizeP[plist_] := Module[{p},
   p[1] := plist[[1]];
   p[n_] := p[n] = plist[[n]] - Sum[ p[i] pInner[p[i], plist[[n]]] /
         pInner[p[i], p[i]], {i, 1, n-1}];
   Table[p[n], {n, 1, Length[plist]}]
  ];
orthonormalizeP[plist_] := Module[{orthog},
   #/Sqrt[pInner[#, #]] &/@orthogonalizeP[plist] (* Normalizing *)
  ];
```

B Legendre Polynomial Generation

```
(* Generate the polynomials in column 1 of table 1 *)
legendrePolynomialList = Simplify[orthogonalizeP[Table[polyGen[n, n, x], {n, 0, 4}]]];

(* Solve for the coefficients by specifying that <Sn,Sn> = 2/(2n+1). So that
  Mathematica will simplify factors of the form a/Sqrt[a^2], add the assumptions
  that each c[i,i] is >0. *)
standardForm =
  Simplify[
   legendrePolynomialList /.
    Solve[
      Table[
       pInner[legendrePolynomialList[[i + 1]], legendrePolynomialList[[i + 1]]]
        == 2 / (2 i + 1), {i, 0, 4}] (* <Sn,Sn> = 2/(2n+1) *)
      ],
      Table[c[i, i], {i, 0, 4}]
     ][[-1]] (* There are multiple solutions with different signs. [[-1]] takes the last
    solution found by Solve[]. *),
   Assumptions → Table[c[i, i] > 0, {i, 0, 4}]
  ];

(* Solve for the coefficients by specifying that <Sn,Sn> = 1. So that
  Mathematica will simplify factors of the form a/Sqrt[a^2], add the assumptions
  that each c[i,i] is >0. *)
normalizedForm = Simplify[
   legendrePolynomialList /.
    Solve[
      Table[
       pInner[legendrePolynomialList[[i + 1]], legendrePolynomialList[[i + 1]]]
        == 1, {i, 0, 4}] (* <Sn,Sn> = 1 *)
      ],
      Table[c[i, i], {i, 0, 4}]
     ][[-1]] (* There are multiple solutions with different signs. [[-1]] takes the last
    solution found by Solve[]. *),
   Assumptions → Table[c[i, i] > 0, {i, 0, 4}]
  ];

(* Display the solutions. *)
Column[{legendrePolynomialList, standardForm, normalizedForm}]
```

$\{c[0,0],\ x\,c[1,1],\ \frac{1}{3}(-1+3x^2)\,c[2,2],\ \frac{1}{5}x(-3+5x^2)\,c[3,3],\ \frac{1}{35}(3-30x^2+35x^4)\,c[4,4]\}$

$\{1,\ x,\ \frac{1}{2}(-1+3x^2),\ \frac{1}{2}x(-3+5x^2),\ \frac{1}{8}(3-30x^2+35x^4)\}$

$\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\,x,\ \frac{1}{2}\sqrt{\frac{5}{2}}(-1+3x^2),\ \frac{1}{2}\sqrt{\frac{7}{2}}\,x(-3+5x^2),\ \frac{3(3-30x^2+35x^4)}{8\sqrt{2}}\}$

C Plot Generation

```
(* Plot the first 13 normalized Legendre polynomials. *)
Plot[Table[LegendreP[n, x] / Sqrt[2 / (2 n + 1)], {n, 0, 12}], {x, -1, 1}]
```