

Simulation of a simple electron gun

David Moore¹

San Diego Miramar College

(Dated: 3 December 2013)

In this paper the properties of a cylindrically symmetric five component electron gun are calculated computationally. The gun consists of a filament, a cathode, an anode, a Wehnelt cylinder/grid, and a focus. The electrostatic potential of the gun is calculated using a laplacian relaxation method and a simple compactification scheme. Electron trajectories are then calculated with relativistic effects taken into account, using the Euler method of integration. Finally, the simulation is compared to experimental results.

I. INTRODUCTION

The electron gun model presented here operates in a vacuum environment. Electrons are released from a hot filament which doubles as the cathode. Surrounding the cathode is the grid or Wehnelt cylinder, which serves to focus the emitted electrons. The electrons then accelerate towards the positively charged (relative to the cathode) anode. Finally, the beam is passed through two more plates which serve to focus the electron beam further. This is illustrated in figure 4.

II. THEORY

A. Model Details

1. *Statement of the problem*

The interaction between the grid, anode, and cathode, is not trivial due to the nature of conductors. One well known method for finding the electric fields around conductors is to solve a boundary value problem. From Gauss's law, one knows that in an area of zero charge density, $\nabla \cdot \vec{E} = 0$. Since $\vec{E} = -\nabla\phi$, this becomes Laplace's equation:

$$\nabla^2\phi = 0 \tag{1}$$

If we specify boundary conditions for this differential equation, namely making sure the potential is zero at infinity and is set to the required voltages on the boundaries of the conducting elements, we end up with a clear task: Given the boundary values, solve equation 1 over all of space.

Because of the cylindrical symmetry, we work exclusively in cylindrical coordinates. We denote such a coordinate by (r, z, θ) , with $r \in (0, \infty)$, $z \in (-\infty, \infty)$, and $\theta \in (0, 2\pi)$. These are mapped to euclidean coordinates by the equation $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$. In these coordinates, equation 1 becomes:

$$0 = \nabla^2\phi = \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} \tag{2}$$

We can take advantage of cylindrical symmetry and recognize that ϕ is independent of θ , so that 2 becomes:

$$0 = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (3)$$

2. Space discretization and compactification

To numerically solve the problem, we first translate the problem to a compact space¹. A compactification is chosen using two scalar parameters a and b , matched to r and z by the equations:

$$z = \frac{b}{1 - b^2} \quad (4)$$

$$r = \frac{a}{1 - a^2} \quad (5)$$

Using equations 3, 4, and 5, we only have to consider the points (a, b) where $a \in [0, 1]$ and $b \in [-1, 1]$. The space of (a, b) is then discretized into an N by M evenly spaced grid. We will denote coordinates of this grid a_{ij} and b_{ij} , where $0 \leq i \leq N$ and $0 \leq j \leq M$. Their corresponding r and z components will be denoted by r_{ij} and z_{ij} , and the value of the potential at these coordinates will be denoted ϕ_{ij} . Note that at $a = \pm 1$ and $b = \pm 1$ equations 4 and 5 become singular. In these cases, standard extended real notation is used to assign values as follows: If $i = 0$, then $a_{ij} = 0$ and $r_{ij} = 0$. If $i = N$, then $a_{ij} = 1$ and $r = \infty$. If $j = 0$, then $b_{ij} = -1$ and $z = -\infty$. Finally, if $j = M$, then $b_{ij} = 1$ and $z = \infty$.

3. Derivative approximations

If a function $f(x)$ has a domain discretized into coordinates x_n , is denoted $f_n = f(x_n)$, and if we denote $\Delta x_n = x_n - x_{n-1}$, then one can immediately deduce three approximations for the first derivative of f .

$$f'_n \approx \frac{f_{n+1} - f_n}{\Delta x_{n+1}} \approx \frac{f_n - f_{n-1}}{\Delta x_n} \approx \frac{f_{n+1} - f_{n-1}}{\Delta x_n + \Delta x_{n-1}} \quad (6)$$

Applying the first part of 6 to f and the second part to the approximation of f' , one finds an approximation for the second derivative.

$$f''_n \approx \frac{f_{n+1} - f_n}{\Delta x_{n+1} \Delta x_n} - \frac{f_n - f_{n-1}}{\Delta x_n^2} \quad (7)$$

4. *Relaxation method*

To solve 3 we use the derivative approximations of the previous section to approximate the differential equation as a difference equation. This equation can then be solved as $\phi_{ij} = F(\phi_{(i+1)j}, \phi_{(i-1)j}, \phi_{i(j+1)}, \phi_{i(j-1)})$. We end up with $N \cdot M$ linear equations. Instead of considering the simultaneous equations, we use the successive over relaxation method as described in *A First Course in Computational Physics*² (p. 262). Briefly, the method works as follows. Initial values for each ϕ_{ij} are chosen. At each step of the relaxation process, ϕ'_{ij} is calculated by solving the difference equation that is a result of equation 3, and is then used as the new value for ϕ_{ij} . Phrased differently, at each step we calculate ϕ'_{ij} as described before and then assign $\phi_{ij} := \phi'_{ij}$. This is the standard relaxation method. The over relaxation method used here assigns $\phi_{ij} := \phi'_{ij} + \alpha(\phi'_{ij} - \phi_{ij})$, where α is a parameter between 0 and 1, in this paper's simulation it was chosen to be $\alpha = .9$. This succeeds in accelerating convergence. Results from this method are presented in the model results section.

5. *Calculating \vec{E} from the potential*

Using the values from the previous section for ϕ_{ij} , one wants to find the $\vec{E} = -\nabla\phi$. This is relatively straightforward by the multivariable chain rule:

$$\begin{aligned}\vec{E} &= -\nabla\phi \\ &= \left(-\frac{\partial}{\partial x}\phi(a(x), b(y)), -\frac{\partial}{\partial y}\phi(a(x), b(y))\right) \\ &= \left(-\frac{\partial a}{\partial x}\frac{\partial}{\partial a}\phi(a, b), -\frac{\partial b}{\partial y}\frac{\partial}{\partial b}\phi(a, b)\right)\end{aligned}\tag{8}$$

6. *Calculating electron trajectories*

Electrons in the model have zero initial velocity and have an initial position at the cathode. The Lorentz force law states that an electron in the absence of a magnetic field has a changing momentum according to:

$$\frac{d\vec{p}}{dt} = q\vec{E}\tag{9}$$

(q is considered here to have a negative value for the electron). Relativistic momentum is given by:

$$\vec{p} = \gamma m \vec{v} \quad (10)$$

where γ is the lorentz factor. From this equation, we can create a vector quantity with units of velocity as $\vec{s} = \frac{\vec{p}}{m}$. Solving 10 for \vec{v} gives:

$$\vec{v} = \vec{s} \left(1 + \left(\frac{s}{c} \right)^2 \right)^{-\frac{1}{2}} \quad (11)$$

($\vec{s} \cdot \vec{s}$ has been denoted by s^2 for notational convenience). Using these equations, we can set up an Euler integration scheme:

$$\vec{x}(t + dt) = \vec{x}(t) + \vec{v}(\vec{s}) dt \quad (12)$$

$$\vec{s}(t + dt) = \vec{s}(t) + \frac{q}{m} \vec{E}(\vec{x}) dt \quad (13)$$

where $\vec{v}(\vec{s})$ is the function given in equation 11. In SI units $\frac{q}{m}$ is on the order of $10^{11} \frac{\text{C}}{\text{kg}}$, which is within the range of double floating point precision, so that no changes need to be made when implementing equations 12 and 13 on a computer while using SI units.

B. Model parameters

Mass to charge ratio of the electron: $\frac{q}{m} = -1.759 \cdot 10^{11} \frac{\text{C}}{\text{kg}}$.

Successive overrelaxation parameter: $\alpha = 0.9$.

C. Model Results

Plots of the model results are shown after the appendix.

D. Comparison with experimental phenomena

One well known property of electron guns is the phenomenon of cutoff. This was successfully recreated in the simulation. It occurs when the grid voltage is low enough that electrons no longer see a force vector towards the anode, and instead see a force vector

away from the grid. This was recreated and pictured in figure 3. Due to the computational complexity of the relaxation, it is difficult to get an accurate value of the cutoff voltage in this model.

E. Conclusion

The model presented suffers from being computationally expensive, and there may be numerical artifacts from the compactification scheme. It would be best to compare the results obtained with this method to results obtained with other compactification schemes. The model may benefit a lot from using a finite element method approach, as even the over relaxation method takes hundreds of iterations to converge to an accurate result, with a grid size of 100 by 200.

REFERENCES

¹In the real number system, a compact space is one that is closed and bounded. In this case the compact space we translate the problem to is all points (a, b) where $a \in [0, 1]$ and $b \in [-1, 1]$.

²P. L. DeVries, *A First Course in Computational Physics* (Jones and Bartlett Publishers, Sudbury, Massachusetts, 2011).

³“An introduction to electron and ion guns,” <http://www.kimphys.com/>, accessed: 2013-12-2.

FIG. 1. 3D plot of the calculated potential

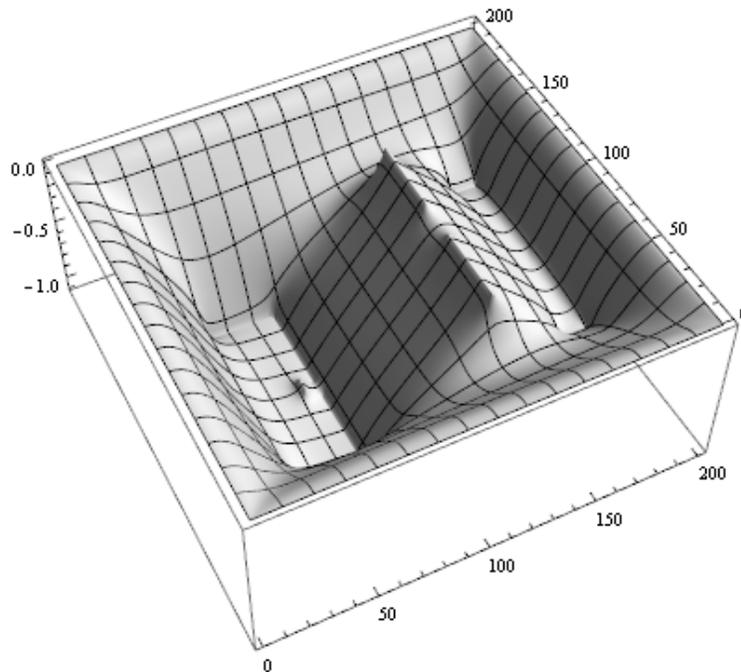


FIG. 2. Plot of calculated electron trajectories

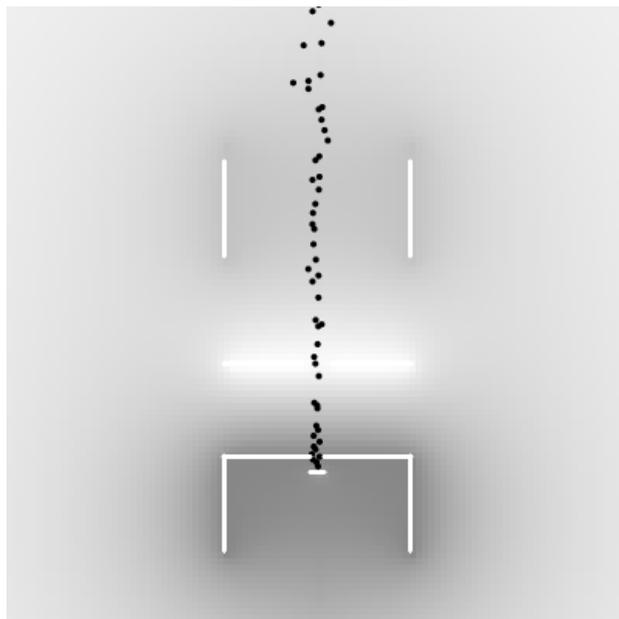


FIG. 3. Plot of calculated electron trajectories in the case of cutoff

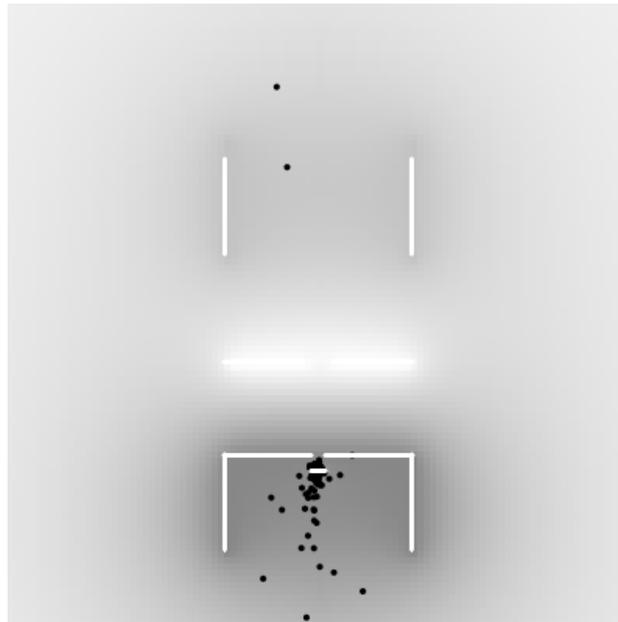


FIG. 4. A schematic of the electron gun³

